The non uniqueness of the space-time energy in General Relativity. The illuminating case of the Schwarzschild metric

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Abstract

We consider the case of asymptotically Minkowskian space-times, showing that, even using rectilinear coordinates in spatial infinity, the energy of such space-times is not uniquely defined. We show this in detail in the case of the Schwarzschild metric whether, or not, its radius source is larger than the Schwarzschild radius, making a supplementary reference to the Reissner-Nordström metric. We explain such an absence of uniqueness in a very natural way and compare it with some known statements and theorems which apparently seem to be in contradiction with it. The suitability of Gauss coordinates when defining a *proper* energy is considered and it is finally concluded, in a natural approach, that a Schwarzschild metric is a particular case of a *creatable* universe.

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I. INTRODUCTION

It has been largely discussed and carefully established that there is a sound definition of energy (and also of linear 3-momentum and angular 4-momentum) of any space-time which is asymptotically Minkowskian, once we have selected a symmetric complex, as for example, the one of Weinberg [1] or the other one of Landau and Lifshitz [2], or some other rather different, but likely equivalent, prescriptions (cf. [3–6] for instance).

Whatever it be, it is generally assumed (see [1] for example) that in order to obtain a sound definition of this energy we must rely on some coordinate system which, fast enough, becomes a rectilinear one in the spatial infinity and then, to simplify the calculation, use the Gauss theorem to write the energy 3-volume integral as a 2-surface integral at this infinity (see [7, 8] for a clear account on this and related topics). Nevertheless, it is obvious, even if not always properly remarked, that this theorem can only be applied if the 3-volume integrand exists and is a continuous function. We will take this fact into account many times along the present paper.

Furthermore, is this sound definition of energy unique? The answer is no, despite of the considerations presented in [1] to justify the opposite (see Appendix A, for a discussion about uniqueness). We will show this in the particular case of a non-rotating, non-charged, black hole, and also in the case of a Schwarzschild metric with a source radius larger than the Schwarzschild radius.

Next, we will see that there is no contradiction with this absence of uniqueness, and finally we will discuss what particular energy can be considered the *proper* energy of this black hole, or of this Schwarzschild metric, showing once more that this absence of uniqueness is not a problem in itself.

Let us recall the expression for the energy, P^0, of an asymptotically Minkowskian metric, $g_{\alpha\beta}$, on the Weinberg complex basis [1]:

$$P^0 = \frac{1}{16\pi}\int \partial_i(\partial_j g_{ij} - \partial_i g_{jj}) d^3x \tag{1}$$

where $i, j, ... = 1, 2, 3$, stand for the 3-space coordinates, $g_{jj} \equiv \delta_{ij} g_{ij}$, $\partial_j g_{ij} \equiv \delta_{jk}\partial_j g_{ik}$, the volume element is $d^3x = dx_1 dx_2 dx_3$, $x_i \equiv \delta_{ij} x^j$, and we have taken the gravitational constant, G, and the speed of light, c, equal to 1. Notice that the Weinberg procedure leads to the same energy expression (1) that the one proposed previously by Arnowitt, Deser and

Misner following a Hamiltonian approach [9] for the Einstein field equations.

Let us now consider the particular case of the Schwarzschild metric in the static standard coordinates:

$$ds^2 = -(1 - \frac{r_0}{r})dt^2 + \frac{dr^2}{1 - \frac{r_0}{r}} + r^2 d\sigma^2, \quad r_0 \equiv 2m, \tag{2}$$

where m is the source mass, and $d\sigma^2 \equiv d\theta^2 + \sin^2\theta \, d\phi^2$ is the metric on the unit 2-sphere. The metric (2) is obviously asymptotically Minkowskian for $r \to \infty$, going like r^{-1} at this distant infinity, and it has a non intrinsic singularity for $r = r_0$, and another intrinsic one for $r = 0$.

If the radius of the spherical source is larger than r_0, this singularity at $r = r_0$ disappears, and because of the Jebsen-Birkhoff theorem [10], the same happens for the intrinsic singularity at $r = 0$ (letting aside particular cases as the one of a black hole surrounded by a spherical non rotating shell at some large radius [11]). Then using Weinberg complex, for example, we can apply Gauss theorem and write (1) as the 2-surface integral on the boundary $r = \infty$

$$P^0 = \frac{1}{16\pi} \lim_{r\to\infty} \int (\partial_j g_{ij} - \partial_i g_{jj}) n_i r^2 d\Omega, \tag{3}$$

where as in (1) the contractions in the 3-space indices are performed with the Kronecker δ_{ij}, and $n_i = x_i/r$, $d\Omega = \sin\theta d\theta d\phi$. As it is well known, applying (3) to the metric (2) in rectilinear coordinates gives the value $P^0 = m$, provided that, as mentioned above, the radius of the spherical source is larger than r_0 (see [1] for example).

But, what is the value of P^0 when (2) represents a black hole so that both metric singularities, at $r = r_0$ and $r = 0$, remain present? To begin with, because of the singularity at $r = r_0$, Gauss theorem cannot be applied to the 3-volume integral in (1), while the partial contribution to this integral, from $r = \infty$ to $r \to r_0$, can be easily seen to diverge in this limit $r \to r_0$.

To overcome this difficulty, we will calculate the *proper* energy P^0 for a black hole, using a convenient family of coordinate systems. We will obtain $P^0 = 0$ (Sec. II), and then we will comment this result and compare it with the other well known result on the subject, $P^0 = m$, when the source radius is larger than r_0 (Sec. III). The remaining sections of the paper are devoted to discuss what could be considered as the *proper* energy of a gravitational field (Sec. IV), and then to apply this notion to establish the creativeness of the Schwarzschild geometry (Sec. V) and, finally, to justify the goodness of the Weinberg complex in the

present issue (Sec. VI). Appendices A and B contain detailed proofs of important results and considerations used to achieve the main results.

A summary of the results of this work has been recently presented at the Spanish Relativity Meeting in Portugal-ERE2012 [12].

II. CALCULATING THE PROPER ENERGY OF A BLACK HOLE

Let us consider the metric of a non rotating, non charged black hole, referred to Lemaître coordinates,

$$ds^2 = -dT^2 + \frac{r_0}{r} dR^2 + r^2 d\sigma^2, \tag{4}$$

(see [2], epigraph 102, or [13]) where the function r defined as

$$r^{3/2} \equiv k(R - \eta T), \quad k \equiv \frac{3}{2}\sqrt{r_0}, \quad \eta \equiv \pm 1, \tag{5}$$

takes all values between 0 and ∞. The Kruskal–Szekeres black and white regions are respectively described with the two coordinate branches provided by $\eta = 1$ and $\eta = -1$ (see [14]). The following considerations apply equally to both (the black and the white) regions, although we shall only refer explicitly to the first one.

We can obtain (4) by making the following change of the t coordinate

$$T = t + \eta\, r_0 f(r), \quad f(r) \equiv 2\sqrt{\frac{r}{r_0}} + \ln\left|\frac{\sqrt{r} - \sqrt{r_0}}{\sqrt{r} + \sqrt{r_0}}\right|, \tag{6}$$

in (2), taking (5) into account.

This time dependent metric (4) is regular everywhere except for the intrinsic singularity $R - \eta T = 0$, corresponding to $r = 0$.

Furthermore, it is written in Gauss coordinates, the metric component $g_{0\alpha}$ being $g_{00} = -1$, $g_{0i} = 0$. This implies that the curves $R = R_0$, $\theta = \theta_0$, $\phi = \phi_0$, with R_0, θ_0 and ϕ_0 constants, are time-like geodesics of the metric. In other words, they describe free-falling particles.

As largely explained in [15] this kind of coordinates is the only one that has to be considered to define a *proper* energy and momenta of a space-time in General Relativity. We will come back next to the question.

But metric (4) does not approach the Minkowski space-time, M_4, for $R \to \infty$ at fixed T, or which is equivalent for $r \to \infty$. This means that the present coordinates (T, R) are not

good coordinates in order to calculate P^0. Then, to reach some good ones, let us change the R coordinate and go to the new coordinate ρ defined as

$$\rho^{3/2} + C = kR, \quad \rho > 0, \tag{7}$$

with C some arbitrary constant.

Then, in the new coordinates (T, ρ, θ, ϕ), the metric (4) becomes:

$$ds^2 = -dT^2 + \frac{\rho}{r} d\rho^2 + r^2 d\sigma^2 \equiv -dT^2 + dl^2, \tag{8}$$

where now r can be written:

$$r^{3/2} = \rho^{3/2} - \eta k T + C. \tag{9}$$

Notice that the 3-metric $dl^2 = \rho/r d\rho^2 + r^2 d\sigma^2$ is flat (its Ricci tensor vanishes) and further, for any given value of $T = T_0$ we can select a C value such that $C = \eta k T_0$, and so such that $r = \rho$, for $T = T_0$, which means that, for any T_0, we can select a corresponding coordinate system where

$$ds^2|_{T_0} \equiv ds^2(T = T_0) = -dT^2 + d\rho^2 + \rho^2 d\sigma^2, \tag{10}$$

that is the Minkowski metric everywhere on the 3-surface $T = T_0$, up to for the essential singularity $r = 0$ $(R = \eta T_0)$, written in spherical coordinates. As a consequence, for each selected T_0, P^0 vanishes,

$$P^0 = 0, \tag{11}$$

according to the definition (1), since in rectilinear x^i coordinates, $x^i x^i = r^2$, dl^2 defined in (8) becomes $dl^2 = \delta_{ij} x^i x^j$. This result, $P^0 = 0$, is obviously true irrespective of the complex used, the one from Weinberg or any other one.

Notice, all the same, the particular algorithm used to define P^0 in the present case: we first select a given space-like 3-surface $\Sigma_3(T_0)$, defined as $T = T_0$, then, by selecting a suitable value of the constant C in (9), we use a particular coordinate system (T, ρ, θ, ϕ) so that, for $T = T_0$, (10) is satisfied, and finally we calculate P^0 for $\Sigma_3(T_0)$ in this suitable coordinate system. As explained, the calculation gives trivially $P^0 = 0$, independently of T_0, provided that for every new T_0 value we suitable change the constant value C. We call this algorithm the *more than quasi-local* algorithm because at the very beginning it involves an integration on a whole 3-volume instead on its boundary and we say next something more on it.

But, why have we not here followed the standard procedure, that is, first calculating P^0 for any T and then particularizing for $T = T_0$? The reason is that (8) presents a physical singularity at $r = 0$. Then, a 3-volume integral like (1) has to be defined first by subtracting an elementary spherical 3-volume, $r = |\epsilon|$, and then taking the limit $|\epsilon| \to 0$. Since no other singularity is present in (8), we can apply the Gauss theorem to this truncated 3-volume integral, and express it like the corresponding flux through the infinite boundary $\rho \to r \to \infty$, plus the contrary flux through $r = |\epsilon|$, before taking finally the limit $|\epsilon| \to 0$. The problem with this hypothetical calculation is that while (8) is asymptotically Minkowskian for $\rho \to \infty$, it is not asymptotically Minkowskian for $r = (\rho^{3/2} - \eta kT + C)^{2/3} \to 0$.

As far as the first limit, $\rho \to \infty$, is concerned, it can be easily seen from (9) that metric (8) goes like $\rho^{-3/2}$ towards Minkowski metric and so faster than ρ^{-1} which is the limit law of decreasing to make sure the convergence of P^0 defined in (3). Let us remark that a proper definition of asymptotically flatness (see for example [7]) should include the suitable kind of asymptotic behavior of the time derivatives of the metric. In our case, we do not need to consider the behavior of these time derivatives since they do not appear neither in (1) nor in (3). They do appear in the corresponding integral expressions of the linear 3-momentum, P^i, and the angular 4-momentum, $J^{\alpha\beta}$, $\alpha, \beta = 0, 1, 2, 3$, of the space-time considered [1]. But, as we will see next, P^i and $J^{\alpha\beta}$ vanish in our case irrespective of the kind of asymptotic behavior, because of the present spherical symmetry.

As far as the second limit is concerned, we cannot use asymptotic rectilinear coordinates at the boundary $|\epsilon| \to 0$. To circumvent this difficulty, we could try to generalize this Minkowskian prescription demanding that in a suitable new Gauss coordinate system the new dl'^2 (see (8)) becomes manifestly conformally flat for $|\epsilon| \to 0$ (in accordance with the ideas exposed in [15] for *universes* which are non asymptotically Minkowskian). But this is not possible, not even for an elementary neighborhood of $T = T_0$, because it can be seen (Appendix B) that the only solution of the vacuum Einstein field equations like

$$ds^2 = -dT'^2 + G(T', \rho')(d\rho'^2 + \rho'^2 d\sigma^2) \tag{12}$$

with $G(T', \rho')$ any regular function of T' and ρ', is just (locally) the Minkowski space-time M_4.

Of course, we could loose this coordinate condition by using Gauss coordinates such that we have (12) only over the boundary 2-surface of the space-like 3-surface $T = T_0$ (this

can always be done: see again [15] and references therein). But this is less than what we have already reached: to have (10), that is (12) with $G(T', \rho') = 1$ everywhere on $T = T_0$. Therefore, the coordinates used to write the Schwarzschild metric in the form (8) are the good (or at least simple good) coordinates to define the *proper* energy P^0 of our black hole, using the so called *more than quasi-local* algorithm, which become $P^0 = 0$.

We discuss in the next section in which sense we can call this vanishing energy of the black hole its *proper* energy. As announced above, we will compare it with the different result $P^0 = m$ for the Schwarzschild metric, concluding that there is no contradiction between these two different results.

But, before leaving the present section, let us say some words about what we have called above "the more than quasi-local" algorithm. Remember that we have been not able to define first P^0 for any time T, in a consistent way (i.e.,using Gauss coordinates which behave appropriately at the suitable boundary), in order to afterwards particularize it for $T = T_0$. Because of this, the algorithm renounces to associate an energy to a time–dependent metric in General Relativity, the energy remaining then associated to any space-like 3-surface, $\Sigma_3(T_0)$, which is less restrictive than the *quasi-local* energy program (for a review, see [16]), where the energy P^0 is namely associated to any closed 2-surface embedded in $\Sigma_3(T_0)$. These considerations explain why we have referred to our algorithm as to a more than quasi-local one.

But the algorithm has the supplementary virtue of having led us to a result, $P^0 = 0$, for appropriate Gauss coordinates, which was to be expected. To see it, notice that the same result for the same kind of coordinates will be obtained (Section III) for a Schwarzschild metric whose source radius is larger than r_0 (a suitable ideal star), this time directly, we mean without using the algorithm. But we could imagine that our ideal star has enough mass as to undergo an ideal collapse preserving the spherical symmetry and without expelling any mass (of course, without radiating any gravitational energy too). In this process, we can hope that P^0 has to remain constant. Thus if initially was $P^0 = 0$, this should be the remaining value when the collapse has been completed, which is just the result obtained using our algorithm.

Explaining it in more general terms, let us accept that our isolated already collapsed black hole has to be a physical system independent of the origin of the physical time T of (8). In other words, it is always the same time–dependent physical system. In particular,

its energy should exhibit no dependence on this origin. Thus, we can select any time T_0 to calculate it. But, this is just what does the *more than quasi-local* algorithm, with the final result that, for (8), the corresponding P^0 does not depend on the free selected T_0, giving precisely $P^0 = 0$.

III. VANISHING ENERGY VERSUS THE MASS ENERGY FOR A SCHWARZSCHILD METRIC

Thus, the energy, P^0, associated to the Schwarzschild metric, (2), in static standard coordinates, takes the value $P^0 = m$, provided that we use the Weinberg complex, for example, and provided that the radius of the spherical source is larger than r_0. In this case, because of the Jebsen–Birkhoff theorem (see [10]) there is no singularity in the metric. Then, changing spherical coordinates in (2) into rectilinear ones x^i, that is $x^i x^i = r^2$, and since we can use the Gauss theorem, we can write the 3-volume integral in (1) as a 2-surface integral on the boundary 2-surface $r \to \infty$, according to (3).

The metric (2) goes like r^{-1} to a Minkowskian metric when $r \to \infty$. This makes very easy the calculation of the above limit which gives the above value $P^0 = m$. The same result is obtained using another complexes in [17].

Then, let us consider the Schwarzschild metric in the form (8) when again the source radius is larger than r_0. This form of the metric has no singularities, and we can apply Gauss theorem, so as to arrive to (3).

But it is very easy to see that, for any given T value, metric (8) approaches like $\rho^{-3/2} \approx r^{-3/2}$ the M_4 metric at the spatial infinity $\rho \to \infty$.

From (3) and this kind of approaching law it is easy to see with rather no calculation that metric (8) gives the value $P^0 = 0$, the same value, by the way, that the one obtained for this metric with a vanishing source radius in the precedent Section, using the *more than quasi-local* algorithm.

This result, $P^0 = 0$, could seem erroneous at a first view since there is a well known statement (cf. [18–21]), for asymptotically Minkowskian spaces, stating, under wide hypothesis including the non-negative character of the local mass energy density, that the only space among these spaces having $P^0 = 0$ is the Minkowski space. But there is no contradiction between our result, $P^0 = 0$, and this theorem, since this one assumes, in particular, that the

3-space metric components, g_{ij}, go to δ_{ij} like r^{-1} when $r \to \infty$, while in our case we have $g_{ij} = \delta_{ij} + O(r^{-3/2})$.

Notice, then, that our result does not mean at all that this theorem, like same sharper ones [22–25] in the literature, were wrong, since we only have been able to circunvent their correct results by changing its plausible boundary conditions, $g_{ij} - \delta_{ij} = O(r^{-1})$ or even $o(r^{-1/2})$, by other different ones, $g_{ij} - \delta_{ij} = O(r^{-3/2})$, unavoidable in a different context: the last boundary conditions are not some ones selected *ad hod* in order to obtain a vanishing universe energy, on the contrary, they come from the physical requirement that in order to define a *proper* energy we should use coordinate systems whose coordinate time was a physical and *universal* time, that is, we should use Gauss coordinate, at least in the elementary neighborhood at the $r \to \infty$ boundary (see next Section). Furthermore, as it was stressed in Ref. [26], our metric falloff, $g_{ij} = \delta_{ij} + O(r^{-3/2})$, and the corresponding vanishing of P^0 are not necessarily in contradiction with the results stablished in the aforementioned references [22–25].

In any case, the fact we want to stress here is that we do not obtain the same value for P^0 for the same space-time (a Schwarzschild metric whose source radius is larger than r_0) when we use static standard coordinates (see Eq. (2)), in which case we obtain $P^0 = m$, than when we use the coordinates (T, ρ, θ, ϕ) of (8), in which case we obtain $P^0 = 0$, though in both cases the metric becomes asymptotically Minkowskian fast enough: like r^{-1} in the first case, and like $r^{-3/2}$ in the second case. Notice that, in any case, this result would not be invalidated by Weinberg's statement in [1] (see epigraph 6, chapter 7), according to which when an asymptotically Minkowskian space-time is referred to different coordinate systems, all them manifestly Minkowskian at the spatial infinity, we always found the same Minkowskian 4-vector for the linear 4-momentum. It would not be invalidated, first because when trying to prove his statement, Weinberg assumes implicitly that both approaches are as fast as r^{-1}, while in our case only one of the two approaches is of this kind, the other going like $r^{-3/2}$; but also because the proof does not seem to completely work (see Appendix A).

IV. PROPER ENERGY AND GAUSS COORDINATES

As it has been already commented in the Introduction, a sound definition of the energy, P^0, of an asymptotically Minkowskian space-time, must rely on coordinate systems which be

rectilinear at the spatial infinity $r \to \infty$. Nevertheless, we have just seen that even in those kind of coordinate systems, a Schwarzschild metric whose source radius is longer than r_0, has different values for P^0 (one of them vanishing) when using different coordinate systems all them becoming fast enough rectilinear at this infinity. Thus, which if any of these boundary rectilinear coordinate systems should be chosen in order to calculate a sound physical energy P^0?

In a series of papers we and other authors [15, 27, 28] have explained why Gauss coordinate systems could be candidates for these preferred coordinate systems. Let us reproduce here some of the arguments used in these references:

To begin with, whatever be the complex used, P^0 is initially expressed as a 3-volume integral, whose integrand is calculated at a given $t = t_0 = constant$, where t is the time coordinate used. In other words, the different elementary contributions to this integral are all them calculated at the same time t_0. But, in order that this equal time has a physical significance, this time has to be a synchronized time, that is, one which gives the same reading for events which are physically simultaneous (we call this a *universal* time), this simultaneity being defined operationally like in Minkowski space-time (on this point we refer the reader to the book [2], epigraph 84) . This *universal* character of time requires the use of coordinates such that the metric components g_{0i} vanish.

On the other hand, if we want to define some sort of physical energy P^0, we should use as a time coordinate a physical (proper or canonical) time, that is we must have $g_{00} = -1$.

All in all, a good coordinate system in order to produce a sound physical energy P^0 would be a Gaussian one.

Furthermore, as it has been pointed in Section II and is well known, a Gauss coordinate system is one which is *adapted* to particles which fall freely in the space-time considered. In other words, their motion equations are $x^i = x^i_0$, where the x^i are the 3-space coordinates and the x^i_0 are three arbitrary constants corresponding to one of these free-falling particles. This *adapted* character of the Gauss coordinates says us that defining P^0 for an asymptotically Minkowskian space-time in these Gauss coordinates can be seen as a sort of generalization to General Relativity of what is called the proper energy of an m_0 mass particle in Minkowski space, which is just the particle energy seen by an instantaneously comoving inertial system, that is m_0. Actually, a suitable family of free-falling non rotating observers is the generalization, when gravitation is present, of an inertial coordinate system

in M_4. Furthermore we will see next that our Schwarzschild metric has vanishing linear and angular 3-momenta in the Gauss coordinates of (8), that is, these coordinates can also been seen as *comoving* coordinates. Thus, when P^0 is calculated in Gaussian coordinates we will call it the *proper* energy of the asymptotically Minkowskian space-time considered, and we will denote it as P^0_p, leaving for the next Section whether this proper energy is unique or not.

The reader could point out that the analogy just established between the proper energy of an m_0 mass particle and P^0_p is misleading, since the first one is m_0, while the second one, P^0_p, for a Schwarzschild metric just vanishes. (Notice that metric (8) corresponds to Schwarzschild metric referred to Gauss coordinates, and so the energy P^0 associated to (8) is really the proper energy, P^0_p, of this metric.) But this difference between both energy values was to be expected, this vanishing being now, when gravitation is present, the net result of the negative gravitational contribution to the total energy of the gravitational field and its source in the particular case of the Schwarzschild metric.

The same type of intuitive explanation can be given to the fact that P^0_p also vanishes for a closed or flat Friedmann-Lemaître-Robertson-Walker (FLRW) universe according to the most literature on the subject (cf. [15, 29, 30]).

As previously announced, let us compare now the two values of P^0 obtained for the energy of a Schwarzschild metric whose radius source is larger that r_0: $P^0_p = 0$, the proper energy, referred to Gauss coordinates (see (8)), and $P^0 = m$ referred to the static standard coordinates (see (2)). These two energies are different because they deal with two different physical situations. In the first case, the total energy is the 3-volume integral of an infinite number of synchronous elementary contributions measured by the corresponding local free-falling observers, while in the second case these observers are prevented to fall by the reaction force derived from some introduced non gravitational energy. Accounting for this non gravitational energy seems the reason why the proper vanishing energy, P^0_p, grows up to the static result $P^0 = m$.

This interpretation also works in the case of the Reissner-Nordström metric (see, for example, [31])

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2 d\sigma^2, \tag{13}$$

where $A(r) \equiv 1 - \frac{2m}{r} + \frac{4\pi Q^2}{r^2}$, with m and Q the source mass and the electric charge, the last

one in unities such that the electric field modulus is Q/r^2.

There is only a (intrinsic) singularity in this metric ($r = 0$) provided that $4\pi Q^2 > m^2$. On the other hand this metric goes like r^{-1} towards the Minkowski metric. Thus we can obtain the corresponding energy P^0 changing to rectilinear coordinates x^i ($x^i x^i = r^2$), applying Gauss theorem, and calculating the corresponding limit in (3). The well known result is again $P^0 = m$, that is the contribution of Q to P^0 vanishes. This result was to be expected from the above interpretation of the same result, $P^0 = m$, for the Schwarzschild case: since the ideal non charged static observers are not attracted by Q, the same energy m is still enough to prevent them to fall in the gravitational field of (13).

Finally, let us come back to the uniqueness question raised above in the present section: given an asymptotically Minkowskian metric, referred to some Gauss coordinate system, does P^0 depend on the particular Gauss system considered? The answer to this question is positive: aside the double result obtained in the Schwarzschild case, $P^0 = m$ or $P^0 = 0$, it is enough to remember that, even a mere Lorentz transformation (any boost, actually) at the infinity $r \to \infty$, will change the P^0 value (see [1], chapter 7, epigraph 6, if necessary). But, then, which, if any, are the good Gauss coordinate systems to be used?

This question will be partially adressed in the next section where we consider the *creativeness* (see [15] and references therein) of the Schwarzschild space-time.

V. CREATIVENESS OF THE SCHWARZSCHILD SPACE-TIME

The same question raised at the end of the precedent Section, can be raised in the more general case of NON asymptotically Minkowskian space-times. The question has been treated in three papers [15, 27, 28] from us and other authors. The partial answer was the following one: when we deal with a *universe* (i.e., a space-time whose well defined linear 4-momentum and angular 4-momentum are conserved), in order to define its proper momenta, we must use coordinates such that:

(a) be Gaussian coordinates,

(b) be such that the 3-space metric is manifestly conformally flat on the 2-surface boundary of the space-like 3-surface $T = T_0$ (that is, for $r \to \infty$, for $T = T_0$),

(c) be such that the corresponding linear 3-momentum, P^i, and angular 3-momentum, J^{ij}, vanish, the last one irrespective of its origin.

Such coordinate systems can be proved to exist for any universe, and we call them *intrinsic* coordinate systems. But, it is still possible that different intrinsic coordinate systems exist for the same universe, leading perhaps to different 4-momenta. Nevertheless, if we find one of these intrinsic coordinate systems such that the corresponding 4-momenta vanish, we must conclude that the proper 4-momenta of this universe vanish in themselves, and then we call it a *creatable* universe. We must conclude this by noticing that Minkowski space is trivially creatable in this precise sense, even if it can be shown that there are intrinsic coordinate systems for which its 4-momenta do not all them vanish [15]. Thus, if a creatable universe has intrinsic coordinate systems whose corresponding 4-momenta do not completely vanish, we must interpret that this is due to the fact that this intrinsic coordinate fails to respect some symmetries of the corresponding universe. Which of them? Just the ones that allow us to find coordinate systems where these two 4-momenta, the linear and the angular ones, vanish. The name *creatable* universe comes from the suggestion that we need this vanishing in order that this universe could arise from a vacuum quantum fluctuation [32, 33].

Applying these ideas to the metric (8) of a black hole, or of a Schwarzschild metric whose source radius is larger than r_0, we first see that because of the spherical symmetry, the above P^i and J^{ij} must vanish. This means that coordinates (T, ρ, θ, ϕ) in (8) satisfy the above property (c). Furthermore, these coordinates are Gaussian coordinates (property (a)), while property (b) is satisfied *a fortiori* since for $r \to \infty$ and $T = T_0$ the corresponding 3-space metric is just manifestly flat. All in all, (T, ρ, θ, ϕ) are an example of *intrinsic* coordinates according to the definition just given. But the same spherical symmetry tell also us that J^{0i}, the mixed components of the angular 4-momentum, vanish too, and since we have seen that P_p^0, the proper energy, vanishes, we must conclude that any non-rotating black hole, or Schwarzschild space with a source radius larger than r_0, are both creatable universes.

On the other hand, a closed FLRW universe, perturbed or not is a creatable universe [27]. However, from what has been concluded in the present paper, it is not clear whether a perturbed non-rotating black hole, i.e., an slightly rotating one, would still be a creatable universe. In other words, the creativeness of a non-rotating black hole could be a non-stable result.

VI. FINAL CONSIDERATIONS: WHY THE WEINBERG COMPLEX?

Let us consider some complex, $\tau^{\alpha\beta}$, in General Relativity, that is $\tau^{\alpha\beta} = \eta^{\alpha\mu}\eta^{\beta\nu}(T_{\mu\nu}+t_{\mu\nu})$, with $T_{\alpha\beta}$ the energy–momentum tensor, $\eta^{\mu\nu}$ the Minkowski tensor and $t_{\alpha\beta}$ some "pseudo–tensor" associated to the presence of the gravitational field, such that the following continuity equation

$$\partial_\alpha \tau^{\alpha\beta} = 0 \tag{14}$$

becomes true. From (14), the following balance equation

$$\frac{d}{dx^0}\int_V \tau^{\alpha 0} dx^3 = -\int_{\Sigma(V)} \tau^{\alpha i} d\Sigma_i, \tag{15}$$

and in particular, the relation

$$\frac{d}{dx^0}\int_V \tau^{00} dx^3 = -\int_{\Sigma(V)} \tau^{0i} d\Sigma_i, \tag{16}$$

follow, the last equation giving the balance between the variation in time of the energy enclosed in a given 3-volume, V, and the flux of this energy trough the boundary 2-surface, $\Sigma(V)$, of this volume.

Given some complex $\tau^{\alpha\beta}$, we can add to it any arbitrary quantity $\partial_\gamma h^{\gamma\alpha\beta}$, such that $h^{\gamma\alpha\beta} = -h^{\alpha\gamma\beta}$, to get another complex

$$\tilde{\tau}^{\alpha\beta} = \tau^{\alpha\beta} + \partial_\gamma h^{\gamma\alpha\beta}, \tag{17}$$

satisfying trivially its own continuity equation

$$\partial_\alpha \tilde{\tau}^{\alpha\beta} = 0, \tag{18}$$

leading to the new balance relation

$$\frac{d}{dx^0}\int_V \tilde{\tau}^{\alpha 0} dx^3 = -\int_{\Sigma(V)} \tilde{\tau}^{\alpha i} d\Sigma_i, \tag{19}$$

as much valid as the original one (15). If we focus our attention on this balance, (15) and (19) are on the same foot, but if what we want is, for example, to calculate P^0, the energy of the corresponding space-time, we are going to obtain, in general, different values for this energy, $\int \tau^{00} d^3x$, or $\int \tilde{\tau}^{00} d^3x$, according to what complex we choose.

Which one, if any, should we chose? Our response is that the Weinberg complex is a specially good candidate for such election, because it comes *directly* from the cornerstone

of the General Relativity building, i.e.,the Einstein field equations. By the word *directly* we mean that the Weinberg complex appears in a (non manifestly covariant) way of writing the Einstein field equations by merely reordering its different terms between both hand sides, without adding any term like $\partial_\gamma h^{\gamma\alpha\beta}$ [1]. Furthermore, in the Introduction we have recalled that the energy P^0 given by (1), from the Weinberg complex, is the same as the Arnowitt-Deser-Misner (ADM) energy [9]. We could then make the conjecture that this is so because the ADM energy comes again from the Einstein field equations, this time written in the standard $3+1$ formalism of the General Relativity (see [7] for an extensive account). In all, this is why, in the present paper, we have used the Weinberg complex in such a preferential way.

On the other hand, the Newtonian gravitational energy per unit mass, ε, is defined from the Newtonian potential φ up to an arbitrary additive constant C, such that $\varepsilon = \varphi + C$. It can be easily seen that this arbitrary constant generates a correction in the post–Newtonian terms of the metric which are by no means deprived of physical effects. Furthermore, let it be the different post–Newtonian solutions to Einstein field equations associated with the different C values. Then, impose suitable physical conditions to these solutions in absence of gravitational radiation, i.e., for $r \to \infty$, the metric becomes manifestly flat (remember that, we can guarantee the existence of $P^0 \neq 0$ by making sure that the metric approaches the Minkowski metric as r^{-1} when $r \to \infty$). In this way we select a unique physical solution of the Einstein field equations to which corresponds the value $C = 0$. Which, again suggest that the above selection of the Weinberg complex, with its *direct* relation to this equations, could be a consistent one.

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Appendix A: Revising a result by Weinberg

In his book [1], epigraph 6 ("Energy, momentum and angular momentum of gravitation"), chapter 7, Weinberg referring to the linear 4-momentum, P^α, of the gravitational field and its sources, states that P^α "have the important property of being invariant under any coordinate transformation that reduces at infinity to the identity". Then he writes such a transformation as

$$x'^\mu = x^\mu + \epsilon^\mu(x) \tag{A1}$$

"where $\epsilon^\mu(x)$ vanishes as $r \to \infty$".

Trying to prove it, Weinberg writes P^λ as

$$P^\lambda = -\frac{1}{8\pi} \lim_{r\to\infty} \int Q^{i0\lambda} n_i r^2 d\Omega, \tag{A2}$$

where quantities $Q^{i0\lambda}$ depend on the space and time derivatives of $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$, and where $\eta_{\mu\nu}$ is the Minkowski tensor. In particular, for P^0 we have (3). Then , the coordinate change (A1) induces an elementary change $\Delta Q^{i0\lambda}$ which can be written as

$$\Delta Q^{i0\lambda} = \partial_j D^{ji0\lambda}, \tag{A3}$$

$D^{ji0\lambda}$ being a linear combination of products of some components of $\partial_\gamma \epsilon^\mu$ and $\eta^{\lambda\rho}$ (see [1] for details), $D^{ji0\lambda}$ being antisymmetric in its j, i indices. This elementary change $\Delta Q^{i0\lambda}$ entails the corresponding elementary change ΔP^λ

$$\Delta P^\lambda = -\frac{1}{8\pi} \lim_{r\to\infty} \int \partial_j D^{ji0\lambda} n_i r^2 d\Omega, \tag{A4}$$

that applying Gauss theorem (assuming that the corresponding 3-volume integrand is a continuous function) gives

$$\Delta P^\lambda = -\frac{1}{8\pi} \int \partial_i \partial_j D^{ji0\lambda} d^3x = 0, \tag{A5}$$

since $D^{ji0\lambda} = -D^{ij0\lambda}$, as the author wanted to prove. In particular $\Delta P^0 = 0$.

The first remark to be made about this reasoning is that we must precise the exact behavior of the above coordinate transformation "that reduce at infinity to identity". To begin with, imagine that matrix $\partial_\gamma \epsilon^\mu$ goes at infinity to zero as the $h_{\mu\nu}$ go. Then, if for $r \to \infty$, $h_{\mu\nu} \to 0$ slower than r^{-1}, integrals (A2) and (A4) would diverge in general, while

with $h_{\mu\nu} \to 0$ fast than r^{-1}, both integrals vanish, and only if $h_{\mu\nu} \to 0$ just as r^{-1} we obtain two finite results.

But, what happens when, as it is the case in our Section III, the transformed components, $h'_{\mu\nu}$, go like $r^{-3/2}$, while the original ones, $h_{\mu\nu}$, go like r^{-1}? This means that the corresponding matrix $\partial_\gamma \epsilon^\mu$ has terms which go like r^{-1} and other ones which go like $r^{-3/2}$, such that finally $h_{\mu\nu}$ and $h'_{\mu\nu}$ behave differently at infinity: as r^{-1} and $r^{-3/2}$, respectively. This double behavior leads to the different results $P^0 = m$ and $P'^0 = 0$ in Section III, showing why in this case the Weinberg proof does not work.

But this is not the only difficulty with the proof. The antisymmetric relation $D^{ji0\lambda} = -D^{ij0\lambda}$, which is essential for the proof, comes from the explicit form $D^{ji0\lambda}$ takes when $r \to \infty$. But the 3-volume integral (A5) deals with $D^{ji0\lambda}$ not only in the infinity 2-surface $r \to \infty$ but all over the 3-volume, where the coordinate transformation $x^\mu \to x'^\mu$ is no more an infinitesimal one as in (A1). Therefore, all over the 3-volume we would have to prove that we still have (A3), with $D^{ji0\lambda} = -D^{ij0\lambda}$. But this does not seem obvious, and any case it is not proved in the Weinberg's reasoning.

All in all, for asymptotically Minkowskian metrics, referred to different asymptotic Lorentzian coordinates, we can have different P^0 values.

Appendix B: Proving the flat-statement

For the sake of completeness, let us consider here a simple proof of the property announced in the text:

Any space-time metric of the form

$$ds^2 = -dT^2 + B(R,T)[dR^2 + R^2(d\theta^2 + \sin^2\theta \, d\phi^2)], \tag{B1}$$

that is a vacuum solution of the Einstein field equations, is necessarily a locally flat metric.

By using the standard $3+1$ formalism of the General Relativity (see, for example, [7]), the notions involved in the proof become very transparent.

The vector field $u = \partial_T$ defines a free-falling radial congruence of observers which, in addition, is vorticity and shear-free. The extrinsic curvature $\mathcal{K}$ of the slices $T =$ constant is proportional to the induced metric γ on these slices, and it is determined by a sole function

Φ,

$$\Phi = -\frac{\dot{B}}{2B} \tag{B2}$$

where the 'dot' stands for the partial derivative with respect to T.

In fact, $\mathcal{K}^R_R = \mathcal{K}^\theta_\theta = \mathcal{K}^\phi_\phi = \Phi$, are the sole non identically vanishing components of $\mathcal{K}$, and then, function Φ is related to the expansion of u,

$$\nabla_\mu u^\mu = -\mathcal{K}^\mu_\mu = -3\Phi. \tag{B3}$$

For a vacuum metric form (B1), the energy flux vanishes and the momentum constraint says that Φ does not depend of R, $\Phi(T)$. The remaining Einstein equations are written as (see, for instance, reference [34]):

$$3\Phi^2 = -\frac{\mathcal{R}}{2}, \tag{B4}$$

$$(\Phi B)^{\cdot} = B\frac{\mathcal{R}}{2} - \frac{F}{R^2} + B\Phi^2, \tag{B5}$$

$$(\Phi B)^{\cdot} = B\frac{\mathcal{R}}{4} + \frac{F}{2R^2} + B\Phi^2, \tag{B6}$$

$\mathcal{R}$ being the scalar curvature of γ, and F is the function

$$F = -\frac{RB'}{4B^2}(RB' + 4B). \tag{B7}$$

where the 'prime' stands for the partial derivative with respect to R.

Taking into account the constraint equation (B4) (energy constraint), the evolution equations (B5) and (B6) are written as:

$$(\Phi)^{\cdot} = \Phi^2, \tag{B8}$$

and

$$F = -BR^2\Phi^2. \tag{B9}$$

The integration is easily accomplished by considering separately the cases $\Phi = 0$ and $\Phi \neq 0$.

(i) If $\Phi = 0$, Eq. (B2) says that B does not depend of T, $B(R)$, and from (B9), $F = 0$. Then, from (B7), $B =$ constant or $BR^4 =$ constant, and the metric form (B1) becomes the Minkowski metric. The last solution is mapped in the standard Minkowski form by performing a radial inversion, $R^* = 1/R$.

(ii) If $\Phi \neq 0$, Eqs. (B2) and (B8) lead to

$$\Phi = -\frac{1}{T+a}, \quad B = f(R)(T+a)^2, \tag{B10}$$

with a an arbitrary constant and $f(R)$ obeying the differential equation

$$f'(Rf' + 4f) = 4f^3 R \tag{B11}$$

as it follows from making compatible Eqs. (B7) and (B9). This equation may be conveniently written as

$$\left(f' + \frac{2f}{R}\right)^2 - \frac{4f^2}{R^2}(1 + R^2 f) = 0, \tag{B12}$$

and then, it is easy to see that the general solution is given by

$$f(R) = \frac{4b^2}{(1 - b^2R^2)^2}, \quad R \in (0, 1/b) \cup (1/b, \infty) \tag{B13}$$

b being an arbitrary constant. Constants a and b are non-essential because they may be absorbed by a trivial redefinition of the coordinates, $\tau = T + a$ and $r = 2bR$.

Therefore, when $\Phi \neq 0$, the sole metric form (B1) that is a solution of the vacuum Einstein equations is the Milne metric:

$$ds^2 = -d\tau^2 + \frac{\tau^2}{(1 - \frac{r^2}{4})^2}[dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)] \tag{B14}$$

that describes a locally flat (Minkowskian) expanding universe.

Notice that the radial inversion $r \to 4/r$ isometrically maps the regions $r < 2$ and $r > 2$ each other.

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